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On the profit of optimizing the fin-keel of a yacht sailing close to wind

A. K. WIERSMA

Dept. of Mathematics, University of Groningen, Groningen, The Netherlands

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SUMMARY
The driving force of a yacht with fin-keel is investigated within the framework of linearized theory. Some configurations to represent fin plus hull are discussed. The driving force produced by a yacht of which the underwater ship has optimum circulation distribution is compared with the driving force produced when the underwater ship has a given planform. In both cases the circulation distribution of the sails is optimized.

1. Introduction

Various authors have described and optimized the action of the sails of a yacht while sailing to windward by considering them as rigid lifting surfaces (e.g. [1], [2], [3]). The water surface is considered to be rigid and flat (zero Froude number). Usually the incoming flow is assumed to have a uniform velocity, however, Milgram [2] approximately took into account the velocity gradient in the boundary layer (near the earth). The flow in which the fin, which under normal sailing conditions always has a small angle of attack, is working seems to be much more complicated. Because the fin terminates at the hull whereas the sails usually do not, the interaction between fin and hull is greater than between sails and hull. De Saix [4] was probably the first to investigate the fin-hull interaction experimentally. Mathematical models with sufficient physical validity are necessarily very complicated and often not accessible to analytical treatment or optimization. Letcher [5] made calculations for hull and keel of the 5.5-meter yacht Antiope assuming a rigid and flat water surface. He used several models to calculate the action of the underwater ship. He considered first the keel alone as a lifting line, and secondly the whole underwater part as a low-aspect-ratio wing. In the third place he used an approach described by Newman and Wu [6] which considers the underwater ship as the combination of a slender body of revolution connected with a slender wing. In [7], the optimum hydrodynamic action of hull plus keel is calculated using a lifting line terminating at the water surface, again for zero Froude number.

In this note we also assume a water surface which is rigid and flat, hoping that for the calculation of sideforce, heeling moment and induced drag this is not too crude. Maybe the best way to describe the action of the hull is one which is also used for large-hub propellers (see e.g. [9]). The hull causes the basic flow of $O(\varepsilon^0)$ in which a lifting surface (the fin) produces disturbances of $O(\varepsilon)$ (ε is the small linearization parameter). Here, however, we consider the hull as a lifting surface obtained by extending the fin to the water surface in one way or another. In Section 4 some configurations are described. Section 5 compares fin-hull combinations of given planform with a combination which would give optimum circulation distribution. In all the cases the circulation distribution of the sails is optimized.

The reason for just investigating the effectiveness of optimizing the underwater ship is the expectation that there will not be much difference between a "regular" and an optimum one. The graphs of the circulation distributions of symmetrical wings show a form not unlike those of the optimum ones (see [7]). Furthermore there is the practical fact that it is much more difficult to optimize the shape of a hull than that of sails.

2. Statement of the problem

We consider a right-handed Cartesian coordinate-system (X, Y, Z) which is fixed to the yacht. The X-axis is parallel to the direction of the undisturbed water velocity V. The centreline of the yacht makes a small angle γ (which is $O(\varepsilon)$, where ε is the small linearization parameter) with the X-axis. The plane $Z = 0$ represents the undisturbed water surface, the region $Z > 0$ the air, $Z < 0$ the water (Fig. 2.1).

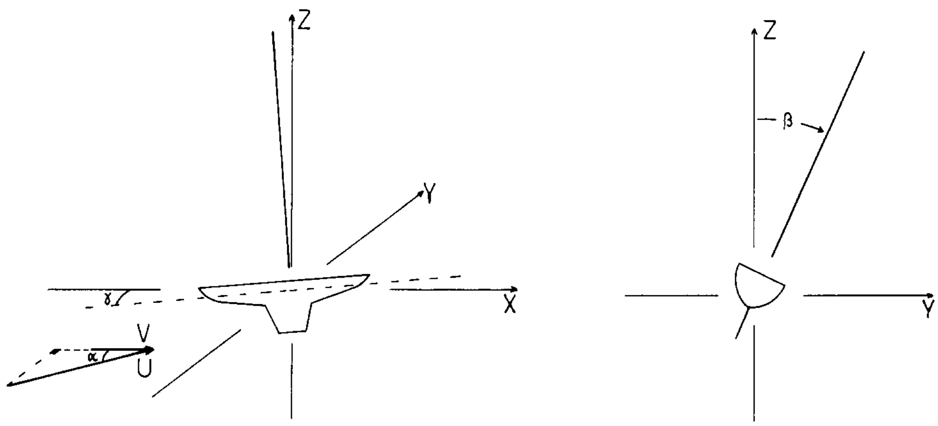


Fig. 2.1. Geometry of the problem.

The air has a uniform velocity U parallel to the plane $Z = 0$ and makes an angle α (also $O(\varepsilon)$) with the X-axis. The yacht will be heeled over an angle β. Without loss of generality the sails can be represented by *one* lifting line. They will have optimum circulation distribution.

Now we want to compute the driving force in a direction which makes an angle of $O(\varepsilon)$ with the X-axis, under the constraints of zero sideforce and zero resultant moment around the X-axis. This moment is caused by the heeling moment of sails and underwater ship and the righting moment of the yacht. In [7] it has been shown that the driving force is the same (up to and including $O(\varepsilon^2)$) in any direction which makes an angle of $O(\varepsilon)$ with the X-axis. This is a consequence of the sideforces of sails and underwater ship being equal in magnitude.

First we consider the case that the geometry of fin and hull is given. For every angle of heel we can calculate lift, moment and induced drag of this combination. This is done by means of lifting-surface theory. The planform by which the underwater ship is represented will in general not yield an optimum circulation distribution. The circulation distribution of the sails is determined such that the conditions for sideforce and moment are fulfilled while their induced drag has its minimum value under the imposed constraints. Then the angle of heel which corresponds with maximum driving force is calculated.

Next both the circulation distribution of sails and underwater ship is optimized. Then the fin-hull combination can also be represented by a lifting line which terminates at the water surface. Again the angle of heel corresponding with maximum driving force is calculated.

We remark that in this problem two linearizations play a role. First we assume that the deviations of sails and fin from surfaces which do not disturb the fluid are small. This allows us to use linearized lifting-surface theory and in the optimization problem lifting-line representation. In the second place the angle between the yacht's course and the apparent wind is small. Therefore we may represent the sails by one lifting line. In both cases the same linearization parameter ε is used.

3. Some necessary formulae

In [7] the case is considered that both sails and underwater ship have optimum circulation distribution. Here we recapitulate the necessary formulae and notations.

Consider a lifting line of length l placed in the plane $X = 0$ of a right-handed Cartesian coordinate system (X, Y, Z) (Fig. 3.1). It is heeled over an angle β and there may exist a gap, defined as the distance between the origin and the line segment.

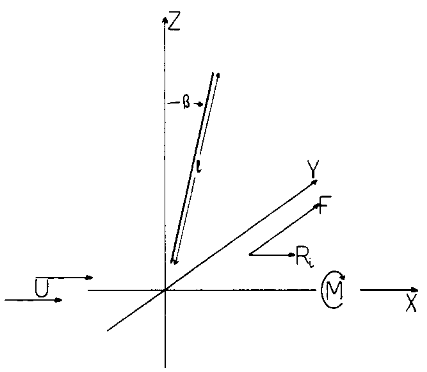


Fig. 3.1. Lifting line and positive direction of forces and moment.

The fluid (incompressible, nonviscous) with density ρ has a uniform velocity U which is parallel to the X-axis. At the plane $Z = 0$ we have the boundary condition of tangential flow. We prescribe the sideforce F and heeling moment M around the X-axis up to and including $O(\varepsilon)$ (Fig. 3.1):

$$F = -\mu_1 \rho U^2 l^2 \cos^2 \beta I_{10}, \quad M = -\mu_2 \rho U^2 l^3 \cos \beta I_{11}. \tag{3.1}$$

The quantities I_{ij} $(i = 1, 2;\ j = 0, 1)$ are functionals of solutions of boundary-value problems. They are described in [7], and are functions of heeling angle β and gap.

From (3.1) the factors μ_1 and μ_2 can be computed when F and M are given. The minimum induced resistance is now:

$$R_i = \tfrac{1}{2}\rho U^2 l^2 \cos^2 \beta \frac{I_{10}}{D} (\mu_1^2 I_{10} I_{21} - 2\mu_1\mu_2 I_{11}^2 + \mu_2^2 I_{11}^2), \tag{3.2}$$

where $D = I_{11}^2 - I_{10}I_{21}$. We observe that R_i is a quadratic function of the prescribed sideforce and moment.

The driving force T in a direction which makes an angle δ of $O(\varepsilon)$ with the X-axis is (up to and including $O(\varepsilon^2)$):

$$T(\varepsilon^2) = \delta F - R_i. \tag{3.3}$$

These formulae are now applied to the case of two coupled lifting lines, one representing the sails, the other one fin plus hull. The velocity of the air is U, that of the water V, and the angle between U and V is α (Fig. 2.1). Quantities belonging to the air are given a superscript a, and for the water we use a superscript w. The balance relations for sideforce and moment are

$$\begin{aligned} &\mu_1 \rho^a U^2 l^{a^2} I_{10}^a = \nu_1 \rho^w V^2 l^{w^2} I_{10}^w, \\ &\nu_2 \rho^a U^2 l^{a^3} \cos\beta I_{11}^a + \mu_2 \rho^w V^2 l^{w^3} \cos\beta I_{11}^w + \rho^w V^2 l^{w^3} m(\beta) = 0, \end{aligned} \tag{3.4}$$

where $m(\beta)$ is the non-dimensional righting moment of the yacht, and ν_1 and ν_2 play the same role for the underwater ship as μ_1 and μ_2 do for the sails. Now μ_1 and μ_2 can be expressed in ν_1 and ν_2 and inserted in the formula for the thrust:

$$T = \alpha F^a - R_i^a - R_i^w. \tag{3.5}$$

T still is a function of ν_1 and ν_2 and can be optimized with respect to these parameters (which means that we vary sideforce and heeling moment until we find the maximum thrust). The optimum thrust becomes

$$T_{\text{opt}} = -\tfrac{1}{2}\rho^w V^2 l^{w^2}[k_1(m(\beta))^2 + k_2\alpha\cos\beta\, m(\beta) + k_3\alpha^2\cos^2\beta] \tag{3.6}$$

where the quantities k_i are given in [7]. They still are functions of heeling angle and gap.

4. The representation of the hull

In this section some configurations for the calculation of sideforce, heeling moment and induced drag produced by the hull are described. The fin is considered as an infinitesimally thin lifting surface. It is extended to the water surface in one way or another in order to represent the hull. The hull we consider here has a waterline length of 7 m and a depth of 0.5 m. Two fins, A and B, are attached separately to this hull. Both fins have a span of 1.2 m and a mean cord of 1.2 m. Hence they have the same lateral area (Figs. 4.1a and 4.1b).

Fin A has a taper ratio (= tip cord/root cord) of 0.27 and the sweep angle Λ of its quarter-cord line is 29°. Fin B has a taper ratio of 0.86 and sweep angle Λ of 35°.

Two hull representations are used for the hull coupled with fin A:

I. a rectangle whose long side is the root cord of fin A,

II. a trapezoid, obtained by approximating the projection of the hull on its centre plane.

The hull coupled with fin B has three representations, two of which are the same as for fin A. A third one is added because it is sometimes used in sideforce calculations:

III. a trapezoid whose oblique sides are obtained by extending the leading and trailing edge of the fin to the water surface.

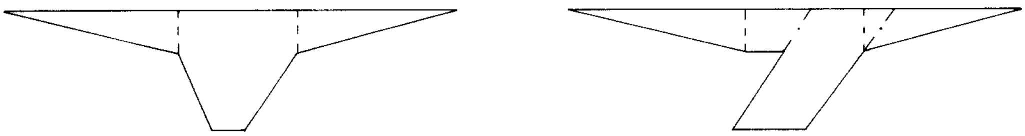


Fig. 4.1. The fin A and B and the distinct hull representations.
– – – –: repr. I, ———: repr. II, –·–·–·–·: repr. III.

In the following we shall write A I if we mean fin A coupled with hull representation I, and analogously A II, B I, B II and B III.

Since by assumption the water surface is rigid and flat, the boundary condition there (i.e. zero normal velocity) is satisfied by considering the fin-hull combination together with its reflection in the water surface. The resulting boundary-value problem is solved with the Vortex-Lattice Method (described e.g. in [8]). Theoretically it gives us lift L and moment M (around the intersection of lifting surface and water surface) accurate up to and including $O(\varepsilon)$. Lift is defined as the force normal to the undisturbed flow. Now we are interested in the driving force T which is of $O(\varepsilon^2)$ in the case of sailing close to wind. Therefore we have to know up to and including $O(\varepsilon^2)$ how far the total hydrodynamic force is "bent backwards" (Fig. 4.2). Hence we calculate the kinetic energy left behind per unit of time by the free vortex sheet. The component R_i of the hydrodynamic force in the direction of the undisturbed water velocity V is now this energy divided by V.

We define lift-, moment- and drag coefficient (C_l, C_m and C_d, respectively) as:

$$C_l = \frac{L}{\frac{1}{2}\rho V^2 l^2 \gamma}, \quad C_m = \frac{M}{\frac{1}{2}\rho V^2 l^3 \gamma}, \quad C_d = \frac{R_i}{\frac{1}{2}\rho V^2 l^2 \gamma^2}, \tag{4.2}$$

where ρ is the density of the fluid, l is the depth of the fin and γ its angle of incidence. Lengths belonging to the underwater ship are nondimensionalized by the depth of the fin l because when optimizing the circulation distribution this is the relevant parameter. We remark that these coefficients still are functions of heeling angle β.

The driving force T in a direction which makes an angle α (of $O(\varepsilon)$) with the hydrofoil is

$$T = L(\alpha + \gamma) - R_i = \tfrac{1}{2}\rho V^2 l^2 \{(C_l - C_d)\gamma^2 + C_l \alpha\gamma\}. \tag{4.3}$$

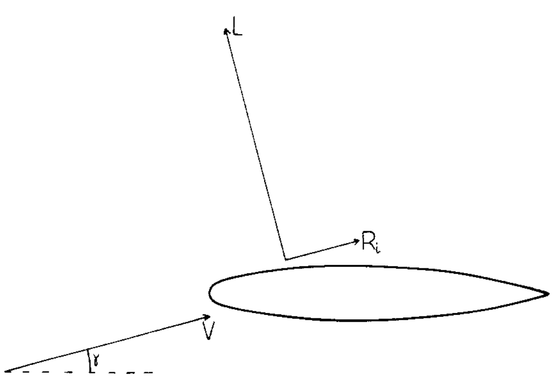


Fig. 4.2. Cross section on a lifting surface and the hydrodynamic forces acting on it.

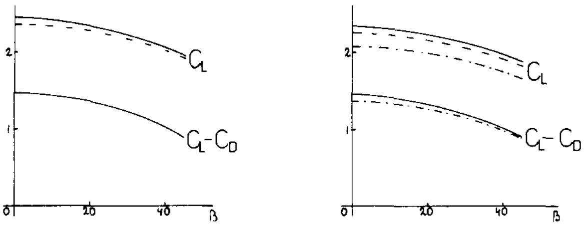


Fig. 4.3. C_l–C_d and C_l for the different fin-hull representations.
----: repr. I, ——: repr. II, –·–·–·: repr. III.

Hence the important quantities for the thrust are C_l–C_d and C_l. These are given in Fig. 4.3a for AI and AII and in Fig. 4.3b for BI, BII and BIII; β ranges from 0° to 45°.

For fin A as well as fin B the values of C_l–C_d for representation I are at most 1% lower than those for representation II. Therefore the graphs of AI and BI are omitted.

5. Thrust production with and without optimized underwater ship

First we derive the optimum thrust for an underwater ship of given planform coupled with optimum sails (see Fig. 2.1). The balance equations for sideforce and moment are found from (3.1), (3.4) and (4.1):

$$\begin{aligned} &-\mu_1\rho^a U^2 l^{a^2} I^a_{10}\cos^2\beta = \tfrac{1}{2}\rho^w V^2 l^{w^2} C_l\gamma\cos^2\beta,\\ &\mu_2\rho^a U^2 l^{a^3} I^a_{11}\cos\beta - \tfrac{1}{2}\rho^w V^2 l^{w^3} C_m\gamma\cos\beta + \rho^w V^2 l^{w^3} m(\beta) = 0. \end{aligned} \tag{5.1}$$

Because of the definition of γ (Fig. 2.1) the angle of incidence of the underwater ship when heeling is $\gamma\cos\beta$. Now μ_1 and μ_2 are known and can be inserted in the formula for the thrust (3.5). We find a quadratic function of γ:

$$T = \tfrac{1}{2}\rho^w V^2 l^{w^2}[C_1 + C_2\gamma + C_3\gamma^2], \tag{5.2}$$

where C_1, C_2 and C_3 still are functions of heeling angle, gap between sails and water surface and α (the angle between the yacht's course and the apparent wind). For fixed α, β and gap the maximum value of T is

$$T_{\max} = \tfrac{1}{2}\rho^w V^2 l^{w^2}\left[C_1 - \tfrac{1}{4}\frac{C_2^2}{C_3}\right] \tag{5.3}$$

which value is obtained for γ equal to

$$\gamma = \tfrac{1}{2}\frac{C_2}{C_3}. \tag{5.4}$$

For a given α and gap we can find by numerical methods the angle of heel which gives maximum thrust.

The formula for the thrust when also the underwater ship has optimum circulation distribution is given in (3.6).

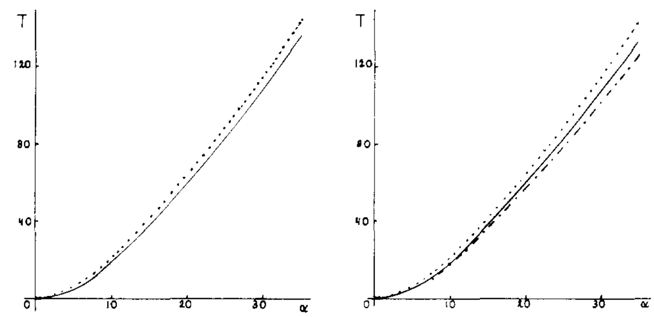
The relevant parameters of the keel and fin of the yacht under consideration are already given in Section 4. Furthermore we need to know:
spanwise length of the sails = l^a = 12 m,
righting moment = $\rho^w V^2 l^{w^3} m(\beta) = 3400 \sin\beta$ kgm,
apparent windspeed = U = 12 m/s,
yacht's speed = V = 3 m/s,
gap between foresail and deck = 1% of l^a = 12 cm.

In Fig. 5.1a we give the optimum thrust (formula 3.6) and the thrust computed for AII (formula 5.3) as a function of α (the angle between relative wind and ship's course). The values for AI are at most 1% smaller than for AII. Fig. 5.1b shows the graphs of the optimum thrust and the driving force of BII and BIII. Again the thrust of BI differs not more than 1% from that of BII.

The thrust T is given in kilograms, angles are given in degrees.

In Fig. 5.2 we give two typical graphs of the heeling angles β for which the maximum thrust is obtained.

Two graphs of the angles of incidence of the underwater ship γ, giving maximum thrust (formula 5.3) are given in Fig. 5.3.



a) fin A b) fin B

Fig. 5.1. The optimum thrust of AII, BII and BIII.
·····: optimal, ——: repr. II, –·–·–·–: repr. III.

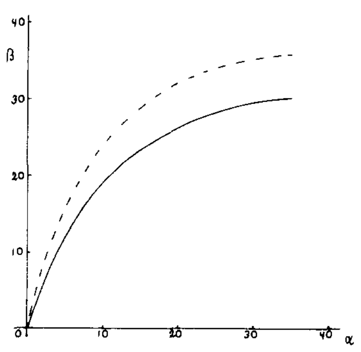


Fig. 5.2. Heeling angles giving maximum thrust.
– – – –: optimum, ——: BIII.

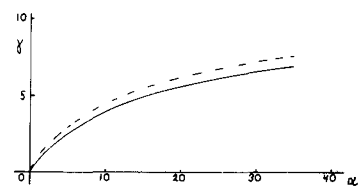


Fig. 5.3. Angles of incidence giving maximum thrust.
– – – –: BIII, ——: AII.

6. Discussion of the results

The mathematical model used here is rather crude: the water surface is rigid and flat, the velocities of wind and water are uniform, the hull is infinitesimally thin and the gap between sails and deck is located at the water surface. However, within this model we use a consistent linearized theory.

As remarked in Section 5 the difference in maximum thrust calculated for fin A with the two hull representations I and II is less than 1%. The difference in maximum thrust for AII and an optimum fin increases until 4% for $\alpha = 35°$.

BI and BII also give practically the same maximum thrust. Here the difference in thrust between BII and an optimum underwater ship increases until 6.5% for $\alpha = 35°$. The calculated maximum thrust for BIII differs at most 10% from the optimum one. It can be expected that the active part of the hull is larger than that taken into account in BIII, so BI or BII may be preferred above BIII.

Although it is doubtful whether the same hull representations have the same validity for fin B as for fin A, we find that AI and AII are systematically about 2.5% better than BI and BII.

There is a tendency (as shown in Fig. 5.2) that fin-hull representations giving a higher driving force also need a larger heeling angle β for which this thrust is obtained. The optimum angle of incidence γ increases when the part of the hull taken into account becomes smaller.

Acknowledgement

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